# STABILITY ESTIMATES FOR THE INVERSE BOUNDARY VALUE PROBLEM FOR THE BIHARMONIC OPERATOR WITH BOUNDED POTENTIALS

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ABSTRACT. In this article, stability estimates are given for the determination of the zeroth-order bounded perturbations of the biharmonic operator when the boundary Neumann measurements are made on the whole boundary and on slightly more than half the boundary, respectively. For the case of measurements on the whole boundary, the stability estimates are of ln-type and for the case of measurements on slightly more than half of the boundary, we derive estimates that are of ln ln-type.

## 1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with $C^\infty$ boundary and consider the following equation:

$$\mathcal{B}_q u := (\Delta^2 + q)u = 0 \text{ in } \Omega, \quad q \in L^\infty(\Omega).$$

Let the domain of $\mathcal{B}_q$ be

$$D(\mathcal{B}_q) := \{u \in H^4(\Omega) : u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\}.$$

We consider the following space for the potential $q$:

$$(1) \qquad \mathcal{Q}_M := \{q : \operatorname{supp}(q) \subset \overline{\Omega}, \text{ and } \|q\|_{L^\infty(\Omega)} \leq M \text{ for some } M > 0\}.$$

We will assume that for all $q \in \mathcal{Q}_M$, 0 is not an eigenvalue for $\mathcal{B}_q$ on the domain $D(\mathcal{B}_q)$. Then given $(f, g) \in H^{7/2}(\partial\Omega) \times H^{3/2}(\partial\Omega)$, there is a unique solution to the boundary value problem:

$$(2) \qquad \mathcal{B}_q u = 0, \quad u|_{\partial\Omega} = f, \quad \Delta u|_{\partial\Omega} = g.$$

The boundary conditions are called Navier conditions [6] and we define the Dirichlet-to-Neumann map $\mathcal{N}_q$ for this operator by

$$\mathcal{N}_q : H^{7/2}(\partial\Omega) \times H^{3/2}(\partial\Omega) \to H^{5/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$$

$$(3) \qquad (f, g) \to (\frac{\partial u}{\partial \nu}|_{\partial\Omega}, \frac{\partial(\Delta u)}{\partial \nu}|_{\partial\Omega}),$$

where $u \in H^4(\Omega)$ is the unique solution to (2).

We are interested in the inverse problem of determining $q$ from $\mathcal{N}_q$. The uniqueness question of determination of $q$ from $\mathcal{N}_q$ was answered in [9, 10] and recently in [11, 12, 17] where they showed that unique determination of both zeroth- and first-order perturbations of the birharmonic operator is

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possible from boundary Neumann data. We note that the papers [11, 17] also show unique determination of the first order perturbation terms from Neumann data measured on possibly small subsets of the boundary.

In this paper, we consider the stability question for the determination of $q$ from $\mathcal{N}_q$ for the operator $\mathcal{B}_q$. That is, whether one can estimate perturbations of $q$ from perturbations of the Neumann data $\mathcal{N}_q$. To the best of the authors' knowledge, stability estimates for inverse problems involving the biharmonic equation has not been obtained earlier, and the purpose of this paper is to investigate it. We prove a stability estimate of ln-type for the case when the Neumann data is measured on the whole boundary. We then prove a stability estimate of ln ln-type when the Neumann data is measured on a part of the boundary that is slightly more than half the boundary.

Our strategy for proving stability estimates follows the methods introduced by Alessandrini in [1] using complex geometric optics (CGO) solutions where a ln-type stability estimate is proved for the Calderón inverse problem [3], and by Heck-Wang in [8] where a ln ln-type stability estimate is proved for the Calderón inverse problem when the Neumann data is measured on slightly more than half of the boundary. CGO solutions were introduced by Sylvester and Uhlmann in the fundamental paper [14] to prove global uniqueness for the Calderón inverse problem. The method in Heck and Wang combines CGO solutions and techniques of [2] with an analytic continuation result of Vessella [16]. Stability estimates for several inverse problems have been obtained in recent years. Apart from the works [1, 8] already mentioned, we refer the reader to [15, 7, 5, 4] for stability estimates involving the Calderón inverse problem and inverse problems involving the Schrödinger or magnetic Schrödinger equation.

## 2. Statements of the main results

We now state the main results of this paper. We first consider stability estimates for full boundary measurements and then prove stability estimates when only partial boundary measurements are available.

2.1. **Results for full boundary measurements.** Consider the following norm on $H^\alpha(\partial\Omega) \times H^\beta(\partial\Omega)$ (for simplicity we will denote this space by $H^{\alpha,\beta}(\partial\Omega)$):

(4) $\qquad \|(f,g)\|_{H^{\alpha,\beta}(\partial\Omega)} = \|f\|_{H^\alpha(\partial\Omega)} + \|g\|_{H^\beta(\partial\Omega)}$ for $(f,g) \in H^{\alpha,\beta}(\partial\Omega)$.

Define:

$$\|\mathcal{N}_q\| = \sup\{\|\mathcal{N}_q(f,g)\|_{H^{\frac{5}{2},\frac{1}{2}}(\partial\Omega)} : \|(f,g)\|_{H^{\frac{7}{2},\frac{3}{2}}(\partial\Omega)} = 1\}$$

where $\mathcal{N}_q(f,g)$ is defined in (3).

**Theorem 2.1.** *Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with smooth boundary. Consider Equation (2) for two potentials $q_1, q_2 \in \mathcal{Q}$. Let $\mathcal{N}_{q_1}$ and $\mathcal{N}_{q_2}$*

*be the corresponding Dirichlet-to-Neumann maps measured on $\partial\Omega$. Then there exists a constant $C = C(\Omega, n, M)$ such that*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)}^2 \leq C \left( \|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| + |\ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\||^{-\frac{4}{n+2}} \right).$$

2.2. **Results for partial boundary measurements.** Now we consider the problem of estimating perturbations of $q$, when the Neumann data $\mathcal{N}_q$ is measured on a subset of $\partial\Omega$ that is slightly more than half of the boundary.

Before stating the result, we introduce the following notation. Let $\alpha \in \mathbb{S}^{n-1}$ be a unit vector and $\epsilon > 0$ be given. Let $\nu(x)$ denote the outer unit normal at $x \in \partial\Omega$. We define

$$(5) \qquad \partial\Omega_{+,\varepsilon} = \{x \in \partial\Omega, \alpha \cdot \nu(x) > \varepsilon\}, \quad \partial\Omega_{-,\varepsilon} = \partial\Omega \setminus \overline{\partial\Omega_{+,\varepsilon}},$$

$$(6) \qquad \partial\Omega_{+} = \{x \in \partial\Omega, \alpha \cdot \nu(x) > 0\}, \quad \partial\Omega_{-} = \partial\Omega \setminus \overline{\partial\Omega_{+}}$$

Now the partial Dirichlet-to-Neumann map is defined as

$$\widetilde{\mathcal{N}}_q : H^{\frac{7}{2}, \frac{3}{2}}(\partial\Omega) \to H^{\frac{5}{2}, \frac{1}{2}}(\partial\Omega_{-,\varepsilon})$$
$$(f, g) \to (\partial_\nu u|_{\partial\Omega_{-,\varepsilon}}, \partial_\nu(\Delta u)|_{\partial\Omega_{-,\varepsilon}}),$$

where $u \in H^4(\Omega)$ is the unique solution to (2). As before, we define the norm of $\widetilde{\mathcal{N}}_q$ as

$$\|\widetilde{\mathcal{N}}_q\| = \sup\{\|\widetilde{\mathcal{N}}_q(f, g)\|_{H^{\frac{5}{2}, \frac{1}{2}}(\partial\Omega_{-,\varepsilon})} : \|(f, g)\|_{H^{\frac{7}{2}, \frac{3}{2}}(\partial\Omega)} = 1\}$$

We have the following stability estimate with partial boundary measurements.

**Theorem 2.2.** *Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with smooth boundary. Consider Equation (2) for two potentials $q_1, q_2 \in \mathcal{Q}$. Let $\widetilde{\mathcal{N}}_{q_1}$ and $\widetilde{\mathcal{N}}_{q_2}$ be the corresponding Dirichlet-to-Neumann maps measured on $\partial\Omega_{-,\varepsilon}$. Then there exist constants $C = C(\Omega, n, M, \varepsilon)$, $K$ and $\theta > 0$ such that*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq \left\{ \|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| + \left( \frac{1}{K} \ln|\ln\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|| \right)^{-\frac{2}{\theta}} \right\}^{\frac{\theta}{2}}.$$

## 3. Preliminary results

We use the following result from [11, 12].

**Proposition 3.1.** [11, Prop. 2.2] *(Interior Carleman estimates) Let $q \in \mathcal{Q}_M$ and $\varphi = x \cdot \alpha$, $|\alpha| = 1$. There exists an $0 < h_0 = h_0(n, M) \ll 1$ and $C = C(n, M) > 0$, where $n$ is the dimension and $M$ is the constant in (1) such that for all $0 < h \leq h_0 \ll 1$ and $u \in C_c^\infty(\Omega)$, we have the following interior estimate:*

$$\|e^{\varphi/h} h^4 \mathcal{B}_q e^{-\varphi/h} u\|_{L^2(\Omega)} \geq \frac{h^2}{C} \|u\|_{H_{scl}^4(\Omega)}.$$

This result is based on a Carleman estimate proven in [13].

**Proposition 3.2.** [11, Prop. 3.2] *(Boundary Carleman estimates) Let $q \in \mathcal{Q}_M$ and $\varphi = x \cdot \alpha$, $|\alpha| = 1$. Let $\partial\Omega_\pm$ be as in (6). There exists an $0 < h_0 = h_0(n, M) \ll 1$ and $C = C(n, M) > 0$, where $n$ is the dimension and $M$ is the constant in (1) such that for all $0 < h \leq h_0 \ll 1$ and $u \in D(\mathcal{B}_q)$, we have the following estimate involving boundary terms:*

$$\|e^{-\varphi/h}h^4 \mathcal{B}_q u\|_{L^2(\Omega)} + h^{3/2}\|\sqrt{-\alpha \cdot \nu}e^{-\varphi/h}\partial_\nu(-h^2 \triangle u)\|_{L^2(\partial\Omega_-)}$$

(7)

$$+ h^{5/2}\|\sqrt{-\alpha \cdot \nu}e^{-\varphi/h}\partial_\nu u\|_{L^2(\partial\Omega_-)} \geq \frac{1}{C}\Big(h^2\|e^{-\varphi/h}u\|_{H^1_{\mathrm{scl}}(\Omega)}$$

$$+ h^{3/2}\|\sqrt{\alpha \cdot \nu}e^{-\varphi/h}\partial_\nu(-h^2 \triangle u)\|_{L^2(\partial\Omega_+)} + h^{5/2}\|\sqrt{\alpha \cdot \nu}e^{-\varphi/h}\partial_\nu u\|_{L^2(\partial\Omega_+)}\Big).$$

Using estimate of Proposition 3.1, the following result is proven in [11, 12] which we will require in what follows.

**Proposition 3.3.** [11, Prop. 2.4] *There exists an $h_0 = h_0(n, M) > 0$ and $C = C(n, M) > 0$, where $n$ is the dimension and $M$ is the constant in (1) such that for all $0 < h \leq h_0 \ll 1$, there exist solutions $u(x, \zeta; h) \in H^4(\Omega)$ to $\mathcal{B}_q u = 0$ in $\Omega$ of the form*

$$u(x, \zeta; h) = e^{\frac{ix \cdot \zeta}{h}}(1 + hr(x, \zeta; h)),$$

*with $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0, |Re(\zeta)| = |Im(\zeta)| = 1$ and $\|r\|_{H^4_{\mathrm{scl}(\Omega)}} \leq Ch^2$.*

We note that the estimates on $h_0$ and $r$ are independent of the potential $q \in \mathcal{Q}_M$.

For proving stability estimates with partial data, we require the following result due to Vessella [16].

**Theorem 3.4.** [16, Theorem 1] *Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected set such that for a positive number $r_0$ the set $\Omega_r = \{x \in \Omega : d(x, \partial\Omega) > r\}$ is connected for every $r \in [0, r_0]$. Let $E \subset \Omega$ be an open set such that $d(E, \partial\Omega) \geq d_0 > 0$. Let $f$ be an analytic function on $\Omega$ with the property that*

$$|D^\alpha f(x)| \leq \frac{C\alpha!}{\lambda^{|\alpha|}} \text{ for } x \in \Omega, \alpha \in (\mathbb{N} \cup \{0\})^n,$$

*where $\lambda, C$ are positive numbers. Then*

$$|f(x)| \leq (2C)^{1-\gamma_1(|E|/|\Omega|)}\left(\sup_E |f(x)|\right)^{\gamma_1(|E|/|\Omega|)},$$

*where $|E|$ and $|\Omega|$ denote the Lebesgue measure of $E$ and $\Omega$ respectively, $\gamma_1 \in (0, 1)$ and $\gamma_1$ depends only on $d_0, \mathrm{diam}(\Omega), n, r_0, \lambda$ and $d(x, \partial\Omega)$.*

We also require the following Green formula:

$$(8) \quad \int_\Omega (\mathcal{B}_q u)\overline{v}\mathrm{d}x - \int_\Omega u\overline{\mathcal{B}_q^* v}\mathrm{d}x = \int_{\partial\Omega} \partial_\nu(\triangle u)\overline{v}\,\mathrm{d}S + \int_{\partial\Omega} \partial_\nu u(\overline{\triangle v})\,\mathrm{d}S$$

$$- \int_{\partial\Omega} (\triangle u)\overline{\partial_\nu v}\,\mathrm{d}S - \int_{\partial\Omega} u(\overline{\partial_\nu(\triangle v)})\,\mathrm{d}S.$$

## 4. STABILITY ESTIMATES WITH FULL BOUNDARY MEASUREMENTS

In this section, we prove Theorem 2.1.

*Proof of Theorem 2.1.* We start with the Green formula (8) and let $q = q_1$ and $u = u_1 - u_2$ and $v \in H^4(\Omega)$ is such that $\mathcal{B}_{q_1}^* v = 0$ in $\Omega$. Here $u_1$ and $u_2$ are solutions to (2) for $q$ replaced by $q_1$ and $q_2$. Then we have

$$(9) \quad \int_\Omega (q_2 - q_1)\, u_2\overline{v}\mathrm{d}x = \int_{\partial\Omega} \partial_\nu(\triangle(u_1 - u_2))\overline{v}\,\mathrm{d}S + \int_{\partial\Omega} \partial_\nu(u_1 - u_2)(\overline{\triangle v})\,\mathrm{d}S.$$

Using Proposition 3.3, we have solutions to $\mathcal{B}_{q_2} u_2 = 0$ and $\mathcal{B}_{q_1}^* v = 0$ of the form

$$(10) \qquad\qquad v(x, \zeta_1; h) = e^{\frac{ix\cdot\zeta_1}{h}}(1 + hr_1(x, \zeta_1; h)),$$

$$(11) \qquad\qquad u_2(x, \zeta_2; h) = e^{\frac{ix\cdot\zeta_2}{h}}(1 + hr_2(x, \zeta_2; h)),$$

where

$$\zeta_1 = \frac{h\xi}{2} + \sqrt{1 - h^2\frac{|\xi|^2}{4}}\beta + i\alpha,$$

$$\zeta_2 = -\frac{h\xi}{2} + \sqrt{1 - h^2\frac{|\xi|^2}{4}}\beta - i\alpha.$$

with $\alpha$ and $\beta$ are unit vectors in $\mathbb{R}^n$ with $\alpha, \beta$ and $\xi$ are mutually perpendicular vectors and $h$ is such that $h \le h_0$ and $1 - h^2\frac{|\xi|^2}{4}$ is positive. Substituting $u_2$ and $v$ into the left hand side of (9), we get,
$$(12)$$
$$\int_\Omega (q_2 - q_1)u_2\overline{v}\mathrm{d}x = \widehat{(q_2 - q_1)}(\xi) + \int_\Omega (q_2 - q_1)\, e^{-ix\cdot\xi}(h\overline{r_1} + hr_2 + h^2\overline{r_1}r_2)\,\mathrm{d}x.$$

Calling the second term on the right hand side of the above equation as $I$, we have the following estimate.

$$|I| \le \int_\Omega |q_2 - q_1|(h|\overline{r}_1| + h|r_2| + h^2|\overline{r}_1||r_2|)\,\mathrm{d}x$$

$$(13) \qquad \le C(h\|r_1\|_{L^2(\Omega)} + h\|r_2\|_{L^2(\Omega)} + h^2\|r_1\|_{L^2(\Omega)}\|r_2\|_{L^2(\Omega)})$$

$$\le Ch \text{ since } h \ll 1.$$

Now consider the right hand side of (9). We have

$$\left|\int_{\partial\Omega} \partial_\nu(\Delta(u_1 - u_2))\bar{v}\mathrm{d}S + \int_{\partial\Omega} \partial_\nu(u_1 - u_2)\overline{(\Delta v)}\mathrm{d}S\right|$$

$$\leq \int_{\partial\Omega} |\partial_\nu(\Delta(u_1 - u_2))\bar{v}|\,\mathrm{d}S + \int_{\partial\Omega} |\partial_\nu(u_1 - u_2)(\overline{\Delta v})|\mathrm{d}S$$

$$\leq \|\partial_\nu(\Delta(u_1 - u_2))\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)} + \|\partial_\nu(u_1 - u_2)\|_{L^2(\partial\Omega)}\|\Delta v\|_{L^2(\partial\Omega)}$$

$$\leq C\big(\|\partial_\nu(\Delta(u_1 - u_2))\|_{L^2(\partial\Omega)}\|v\|_{H^1(\Omega)} + \|\partial_\nu(u_1 - u_2)\|_{L^2(\partial\Omega)}\|\Delta v\|_{H^1(\Omega)}\big)$$

$$\leq C\big(\|\partial_\nu(\Delta(u_1 - u_2))\|_{L^2(\partial\Omega)} + \|\partial_\nu(u_1 - u_2)\|_{L^2(\partial\Omega)}\big)\big(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}\big)$$

$$\leq C\big(\|\partial_\nu(\Delta(u_1 - u_2)), \partial_\nu(u_1 - u_2)\|_{H^{\frac{1}{2},\frac{5}{2}}(\partial\Omega)}\big)\big(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}\big).$$

which again is

$$= C\|(\mathcal{N}_{q_1} - \mathcal{N}_{q_2})(f,g)\|_{H^{\frac{5}{2},\frac{1}{2}}(\partial\Omega)}\big(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}\big)$$

$$\leq C\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|\|(f,g)\|_{H^{\frac{7}{2},\frac{3}{2}}(\partial\Omega)}\big(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}\big)$$

$$\leq C\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|\big(\|u_2\|_{H^4(\Omega)} + \|\Delta u_2\|_{H^2(\Omega)}\big)\big(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)}\big).$$

We have the following estimates for $\|v\|_{H^1(\Omega)}$ and $\|\Delta v\|_{H^1(\Omega)}$. In these estimates, we use that $\Omega \subset B(0,R)$ for $R > 0$ fixed. Then $|e^{\frac{ix\cdot\zeta_j}{h}}| \leq e^{\frac{2R}{h}}$, since $|\zeta_j| = 2$ for $j = 1, 2$.

$$\|v\|_{H^1(\Omega)} \leq \|e^{\frac{ix\cdot\zeta_1}{h}}(1 + hr_1)\|_{L^2(\Omega)} + \sum_{k=1}^{n}\|he^{\frac{ix\cdot\zeta_1}{h}}\partial_{x_k}r_1 + \frac{i}{h}\zeta_{1k}e^{\frac{ix\cdot\zeta_1}{h}}(1 + hr_1)\|_{L^2(\Omega)}$$

$$\leq e^{\frac{2R}{h}}\big(1 + h\|r_1\|_{H^4_{\mathrm{scl}}(\Omega)}\big) + \sum_{k=1}^{n}\left(2e^{\frac{2R}{h}}\left(\|r_1\|_{H^4_{\mathrm{scl}}(\Omega)} + \frac{1}{h}\right)\right)$$

$$\leq Ce^{\frac{2R}{h}}(1 + h^2) + \frac{C}{h}e^{\frac{2R}{h}}(1 + h^2) \leq \frac{C}{h}e^{\frac{2R}{h}}.$$

From straightforward computations, we have the following:

$$\Delta v = he^{\frac{ix\cdot\zeta_1}{h}}\Delta r_1 + 2ie^{\frac{ix\cdot\zeta_1}{h}}(\zeta_1 \cdot \nabla r_1)$$

$$\partial_{x_j}(\Delta v) = he^{\frac{ix\cdot\zeta_1}{h}}\partial_{x_j}(\Delta r_1) + i\zeta_{1j}e^{\frac{ix\cdot\zeta_1}{h}}\Delta r_1 + 2ie^{\frac{ix\cdot\zeta_1}{h}}\partial_{x_j}(\zeta_2 \cdot \nabla r_1)$$

$$\qquad - \frac{2}{h}\zeta_{1j}e^{\frac{ix\cdot\zeta_1}{h}}(\zeta_1 \cdot \nabla r_1).$$

$$\partial_{x_k}\partial_{x_j}(\Delta u_2) = he^{\frac{ix\cdot\zeta_2}{h}}\partial_{x_k}\partial_{x_j}(\Delta r_2) + i\zeta_{2k}e^{\frac{ix\cdot\zeta_2}{h}}\partial_{x_j}(\Delta r_2) + i\zeta_{2j}e^{\frac{ix\cdot\zeta_2}{h}}\partial_{x_k}(\Delta r_2)$$

$$\qquad - \frac{1}{h}\zeta_{2j}\zeta_{2k}e^{\frac{ix\cdot\zeta_2}{h}}\Delta r_2 + 2ie^{\frac{ix\cdot\zeta_2}{h}}\partial_{x_k}\partial_{x_j}(\zeta_2 \cdot \nabla r_2)$$

$$\qquad - \frac{2}{h}\zeta_{2k}e^{\frac{ix\cdot\zeta_2}{h}}\partial_{x_j}(\zeta_2 \cdot \nabla r_2) - \frac{2}{h}\zeta_{2j}e^{\frac{ix\cdot\zeta_2}{h}}\partial_{x_k}(\zeta_2 \cdot \nabla r_2)$$

$$\qquad - \frac{2i}{h^2}\zeta_{2j}\zeta_{2k}e^{\frac{ix\cdot\zeta_2}{h}}(\zeta_2 \cdot \nabla r_2).$$

$$\partial_{x_m}\partial_{x_l}\partial_{x_k}\partial_{x_j}u_2 = he^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_m}\partial_{x_l}\partial_{x_k}\partial_{x_j}r_2 + \mathrm{i}\zeta_{2m}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_l}\partial_{x_k}\partial_{x_j}r_2$$

$$+ \mathrm{i}\zeta_{2l}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_m}\partial_{x_k}\partial_{x_j}r_2 - \frac{1}{h}\zeta_{2l}\zeta_{2m}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_k}\partial_{x_j}r_2$$

$$- \frac{1}{h}\zeta_{2m}\zeta_{2k}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_l}\partial_{x_j}r_2 + \mathrm{i}\zeta_{2k}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_m}\partial_{x_l}\partial_{x_j}r_2$$

$$- \frac{1}{h}\zeta_{2l}\zeta_{2k}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_m}\partial_{x_j}r_2 - \frac{\mathrm{i}}{h^2}\zeta_{2m}\zeta_{2l}\zeta_{2k}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_j}r_2$$

$$- \frac{1}{h}\zeta_{2m}\zeta_{2j}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_l}\partial_{x_k}r_2 + \mathrm{i}\zeta_{2j}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_m}\partial_{x_l}\partial_{x_k}r_2$$

$$- \frac{1}{h}\zeta_{2l}\zeta_{2j}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_m}\partial_{x_k}r_2 - \frac{\mathrm{i}}{h^2}\zeta_{2m}\zeta_{2l}\zeta_{2j}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_k}r_2$$

$$- \frac{\mathrm{i}}{h^2}\zeta_{2m}\zeta_{2j}\zeta_{2k}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_l}r_2 - \frac{1}{h}\zeta_{2j}\zeta_{2k}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_m}\partial_{x_l}r_2$$

$$- \frac{\mathrm{i}}{h^2}\zeta_{2l}\zeta_{2j}\zeta_{2k}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}\partial_{x_m}r_2 + \frac{1}{h^4}\zeta_{2m}\zeta_{2l}\zeta_{2j}\zeta_{2k}e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}(1 + hr_2).$$

Now using the above derivatives, it is straightforward to show the following:

$$(14) \qquad \|\Delta v\|_{H^1(\Omega)} \le \frac{Ce^{\frac{2R}{h}}}{h^2}\|r_1\|_{H^4_{\mathrm{scl}}(\Omega)} \le \frac{C}{h}e^{\frac{2R}{h}}.$$

$$(15) \qquad \|\Delta u_2\|_{H^2(\Omega)} \le \frac{C}{h}e^{\frac{2R}{h}}.$$

$$(16) \qquad \|u_2\|_{H^4(\Omega)} \le \frac{C}{h^4}e^{\frac{2R}{h}}.$$

Therefore we have

$$\left|\int_{\partial\Omega}\partial_\nu(\Delta(u_1 - u_2))\bar{v}\,\mathrm{d}S + \int_{\partial\Omega}\partial_\nu(u_1 - u_2)\overline{(\Delta v)}\,\mathrm{d}S\right|$$

$$\le C\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|(\frac{C}{h^4}e^{\frac{2R}{h}} + \frac{C}{h}e^{\frac{2R}{h}})(\frac{C}{h}e^{\frac{2R}{h}} + \frac{C}{h}e^{\frac{2R}{h}})$$

$$\le \frac{C}{h^4}e^{\frac{2R}{h}}.\frac{C}{h}e^{\frac{2R}{h}}\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| \le \frac{C}{h^5}e^{\frac{4R}{h}}\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|.$$

Now using the fact that $\frac{1}{h} \le e^{\frac{R}{h}}$, we obtain

$$\left|\int_{\partial\Omega}\partial_\nu(\Delta(u_1 - u_2))\bar{v}\,\mathrm{d}S + \int_{\partial\Omega}\partial_\nu(u_1 - u_2)\overline{(\Delta v)}\,\mathrm{d}S\right| \le Ce^{\frac{9R}{h}}\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|.$$

Extending $q_1, q_2$ to $\mathbb{R}^n$ by 0 and using (12) and (13), we get the estimate

$$|\widehat{(q_1 - q_2)}(\xi)| \le C(e^{\frac{9R}{h}}\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| + h)$$

Now

$$\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \le \|q_1 - q_2\|^2_{H^{-1}(\mathbb{R}^n)}$$

$$= \int_{|\xi|\le\rho}\frac{|\widehat{(q_1 - q_2)}(\xi)|^2}{1 + |\xi|^2}\,\mathrm{d}\xi + \int_{|\xi|>\rho}\frac{|\widehat{(q_1 - q_2)}(\xi)|^2}{1 + |\xi|^2}\,\mathrm{d}\xi,$$

for appropriate $\rho$ to be chosen later.
But

$$\int_{|\xi|>\rho} \frac{|\widehat{(q_1-q_2)}(\xi)|^2}{1+|\xi|^2}\, d\xi \leq \int_{|\xi|>\rho} \frac{|\widehat{(q_1-q_2)}(\xi)|^2}{1+\rho^2}\, d\xi$$

$$\leq \frac{1}{\rho^2}\|q_1-q_2\|^2_{L^2(\mathbb{R}^n)} \leq \frac{C}{\rho^2}$$

and

$$\int_{|\xi|\leq\rho} \frac{|\widehat{(q_1-q_2)}(\xi)|^2}{1+|\xi|^2}\mathrm{d}\xi \leq C \int_{|\xi|\leq\rho} \frac{(e^{\frac{9R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|+h)^2}{1+|\xi|^2}\mathrm{d}\xi$$

$$\leq C(e^{\frac{18R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2+h^2)\int_{|\xi|\leq\rho}\frac{d\xi}{1+|\xi|^2}$$

$$\leq C\rho^n(e^{\frac{18R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2+h^2)$$

Therefore

$$\|q_1-q_2\|^2_{H^{-1}(\Omega)} \leq C\rho^n e^{\frac{18R}{h}}\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|^2+C\rho^n h^2+\frac{C}{\rho^2}\ .$$

Now assume that $\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\| < \delta = e^{-\frac{20R}{h_0}}$. Then we choose $\rho = \{\frac{1}{20R}|\ln\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|\|\}^{\frac{2}{n+2}}$. Further let $h = \frac{1}{\rho^{\frac{n+2}{2}}}$. With this choice of $h$, we show that $h < h_0$ and $1-h^2|\xi|^2/4 > 0$ for $|\xi| < \rho$. The fact that $h < h_0$ follows from these inequalities:

$$\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\| < e^{-\frac{20R}{h_0}} \ll 1$$

$$\Rightarrow \ln\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\| < -\frac{20R}{h_0}$$

$$\Rightarrow |\ln\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|\| > \frac{20R}{h_0}$$

$$\Rightarrow \frac{1}{20R}|\ln\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|\| > \frac{1}{h_0}$$

$$\Rightarrow h < h_0.$$

Now we show that $1-h^2\frac{|\xi|^2}{4} > 0$ for $|\xi| < \rho$. We have that

$$\rho^n = \{\frac{1}{20R}|\ln\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|\|\}^{\frac{2n}{n+2}}.$$

Since $\frac{2n}{n+2} > 1$ and $\frac{1}{20R}|\ln\|\mathcal{N}_{q_1}-\mathcal{N}_{q_2}\|\| > 1$, we have that $\rho^n > 1$.
Hence

$$h^2\frac{|\xi|^2}{4} < h^2\frac{\rho^2}{4} = \frac{1}{4\rho^n} < 1$$

and so $1-h^2\frac{|\xi|^2}{4} > 0$.

Therefore

$$\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \le \frac{C}{h^{\frac{2n}{n+2}}} e^{\frac{18R}{h}} \|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|^2 + Ch^{\frac{4}{n+2}}$$
$$\le Ce^{\frac{20R}{h}} \|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|^2 + Ch^{\frac{4}{n+2}}.$$

and since $\frac{1}{h} = \frac{1}{20R}|\ln \|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\||$, we then obtain the estimate

$$\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \le C(\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| + |\ln\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\||^{-\frac{4}{n+2}}),$$

when $\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| < \delta = e^{-\frac{20R}{h_0}}$.

The case when $\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| \ge \delta$ follows from the continuous inclusions

$$L^\infty(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega).$$

In other words, we have

$$\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \le C\|q_1 - q_2\|^2_{L^\infty(\Omega)} \le \frac{4CM^2}{\delta}\delta \le \frac{4CM^2}{\delta}\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|$$

and hence the desired estimate follows. This concludes the proof.    $\square$

## 5. STABILITY ESTIMATE FOR SLIGHTLY MORE THAN HALF DATA

Here we prove stability estimates for the partial data case. In the appendix, we include a proof of the identifiability in this case using linear Carleman weights. We would be using a few estimates derived therein in this section.

*Proof of Theorem 2.2.* We begin with the following identity as at the beginning of Theorem 2.1 and rewrite it as

$$\int_\Omega (q_2 - q_1)\, u_2 \overline{v} \mathrm{d}x = \int_{\partial\Omega} \partial_\nu(\triangle(u_1 - u_2))\overline{v}\, \mathrm{d}S + \int_{\partial\Omega} \partial_\nu(u_1 - u_2)(\overline{\triangle v})\, \mathrm{d}S$$

$$(17) \qquad = \int_{\partial\Omega_{-,\varepsilon}} \partial_\nu(\triangle(u_1 - u_2))\overline{v}\, \mathrm{d}S + \int_{\partial\Omega_{-,\varepsilon}} \partial_\nu(u_1 - u_2)(\overline{\triangle v})\, \mathrm{d}S$$

$$(18) \qquad + \int_{\partial\Omega_{+,\varepsilon}} \partial_\nu(\triangle(u_1 - u_2))\overline{v}\, \mathrm{d}S + \int_{\partial\Omega_{+,\varepsilon}} \partial_\nu(u_1 - u_2)(\overline{\triangle v})\, \mathrm{d}S.$$

We estimate the terms in (17). Proceeding as with the full data case, we have

$$|\int_{\partial\Omega_{-,\epsilon}} \partial_\nu(\Delta(u_1 - u_2))\bar{v} \, \mathrm{d}S + \int_{\partial\Omega_{-,\epsilon}} \partial_\nu(u_1 - u_2)\overline{(\Delta v)} \, \mathrm{d}S|$$

$$\leq C \left( \|\partial_\nu(\Delta(u_1 - u_2))\|_{H^{\frac{1}{2}}(\partial\Omega_{-,\epsilon})} + \|\partial_\nu(u_1 - u_2)\|_{H^{\frac{5}{2}}(\partial\Omega_{-,\epsilon})} \right)$$

$$(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)})$$

$$= C\|(\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2})(f,g)\|_{H^{\frac{5}{2},\frac{1}{2}}(\partial\Omega_{-,\epsilon})}(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)})$$

$$\leq C\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|\|(f,g)\|_{H^{\frac{7}{2},\frac{3}{2}}(\partial\Omega)}(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)})$$

$$\leq C\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|(\|u_2\|_{H^4(\Omega)} + \|\Delta u_2\|_{H^2(\Omega)})(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{H^1(\Omega)})$$

$$\leq Ce^{\frac{9R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|.$$

Now we estimate the terms in (18).

We first have

$$|\int_{\partial\Omega_{+,\epsilon}} \partial_\nu(\Delta(u_1 - u_2))\bar{v} \, \mathrm{d}S| = |\int_{\partial\Omega_{+,\epsilon}} e^{-\frac{x\cdot\alpha}{h}} \partial_\nu(\Delta(u_1 - u_2))e^{\frac{x\cdot\alpha}{h}}\bar{v} \, \mathrm{d}S|$$

$$\leq \|e^{-\frac{x\cdot\alpha}{h}} \partial_\nu(\Delta(u_1 - u_2))\|_{L^2(\partial\Omega_{+,\epsilon})}\|e^{\frac{x\cdot\alpha}{h}}\bar{v}\|_{L^2(\partial\Omega_{+,\epsilon})}$$

$$\leq C\|e^{-\frac{x\cdot\alpha}{h}} \partial_\nu(\Delta(u_1 - u_2))\|_{L^2(\partial\Omega_{+,\epsilon})}$$

and

$$|\int_{\partial\Omega_{+,\epsilon}} \partial_\nu(u_1 - u_2)\overline{(\Delta v)} \, \mathrm{d}S| = |\int_{\partial\Omega_{+,\epsilon}} e^{-\frac{x\cdot\alpha}{h}} \partial_\nu(u_1 - u_2)e^{\frac{x\cdot\alpha}{h}}\overline{(\Delta v)} \, \mathrm{d}S|$$

$$\leq \|e^{-\frac{x\cdot\alpha}{h}} \partial_\nu(u_1 - u_2)\|_{L^2(\partial\Omega_{+,\epsilon})}\|e^{\frac{x\cdot\alpha}{h}}\overline{(\Delta v)}\|_{L^2(\partial\Omega_{+,\epsilon})}$$

$$\leq C\|e^{-\frac{x\cdot\alpha}{h}} \partial_\nu(u_1 - u_2)\|_{L^2(\partial\Omega_{+,\epsilon})}.$$

By the boundary Carleman estimate, we have for $\varepsilon > 0$,

$$h^{\frac{3}{2}}\|\sqrt{\alpha\cdot\nu} \, e^{-\frac{x\cdot\alpha}{h}} \, \partial_\nu(-h^2\Delta u)\|_{L^2(\partial\Omega_+)} \leq C(\|e^{-\frac{x\cdot\alpha}{h}} (h^4\mathcal{B}_{q_1})u\|_{L^2(\Omega)} +$$

$$h^{\frac{3}{2}}\|\sqrt{-\alpha\cdot\nu} \, e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(-h^2\Delta u)\|_{L^2(\partial\Omega_-)} + h^{\frac{5}{2}}\|\sqrt{-(\alpha\cdot\nu)} \, e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_-)})$$

We then have

$$\|\sqrt{\alpha\cdot\nu} \, e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta u)\|_{L^2(\partial\Omega_+)} \leq C\left(\sqrt{h}\|e^{-\frac{x\cdot\alpha}{h}}\mathcal{B}_{q_1}u\|_{L^2(\Omega)}\right.$$

$$+ \frac{1}{h}\|\sqrt{-\alpha\cdot\nu} \, e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_-)}$$

$$\left. + \|\sqrt{-\alpha\cdot\nu} \, e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta u)\|_{L^2(\partial\Omega_-)}\right).$$

Since on $\partial\Omega_{+,\varepsilon}$, $\alpha \cdot \nu > \varepsilon$, we have

$$\|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta u)\|_{L^2(\partial\Omega_{+,\epsilon})} \leq \frac{C}{\sqrt{\epsilon}}\Big(\sqrt{h}\|e^{-\frac{x\cdot\alpha}{h}}\mathcal{B}_{q_1}u\|_{L^2(\Omega)} + \frac{1}{h}\|\sqrt{-(\alpha\cdot\nu)}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_-)}$$
$$+ \|\sqrt{-(\alpha\cdot\nu)}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta u)\|_{L^2(\partial\Omega_-)}\Big).$$
$$\leq \frac{C}{\sqrt{\varepsilon}}\Big(\sqrt{h}\|e^{-\frac{x\cdot\alpha}{h}}\mathcal{B}_{q_1}u\|_{L^2(\Omega)}$$
$$+ \sqrt{-\inf_{\partial\Omega_-}(\alpha\cdot\nu)}\|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta u)\|_{L^2(\partial\Omega_{-,\epsilon})}$$
$$+ \sqrt{-\inf_{\partial\Omega_-}(\alpha\cdot\nu)}\|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_{-,\epsilon})}\Big).$$

Therefore,

$$|\int_{\partial\Omega_{+,\epsilon}}\partial_\nu(\Delta(u_1-u_2))\bar{v}\ \mathrm{d}S| \leq \frac{C}{\sqrt{\epsilon}}\sqrt{-\inf_{\partial\Omega_-}(\alpha\cdot\nu)}\ \|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta(u_1-u_2))\|_{L^2(\partial\Omega_{-,\epsilon})}$$
$$+ C\sqrt{\frac{h}{\epsilon}}\|e^{-\frac{x\cdot\alpha}{h}}(\mathcal{B}_{q_1})(u_1-u_2)\|_{L^2(\Omega)}$$
$$+ \frac{C}{h\sqrt{\epsilon}}\sqrt{-\inf_{\partial\Omega_-}(\alpha\cdot\nu)}\ \|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(u_1-u_2)\|_{L^2(\partial\Omega_{-,\epsilon})}.$$

Now we have

$$\|e^{-\frac{x\cdot\alpha}{h}}\mathcal{B}_{q_1}(u_1-u_2)\|_{L^2(\Omega)} = \|e^{-\frac{x\cdot\alpha}{h}}(q_1-q_2)u_2\|_{L^2(\Omega)}.$$

Using the CGO solutions from Proposition (3.3),

$$u_2(x,\zeta_2;h) = e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}(1+hr_2(x,\zeta_2;h))\ \text{where}$$
$$\zeta_2 = -\frac{h\xi}{2} + \sqrt{1-h^2\frac{|\xi|^2}{4}}\beta - \mathrm{i}\alpha,$$

we have

$$\|e^{-\frac{x\cdot\alpha}{h}}(q_1-q_2)u_2\|_{L^2(\Omega)} \leq C.$$

Therefore

$$|\int_{\partial\Omega_{+,\epsilon}}\partial_\nu(\Delta(u_1-u_2))\bar{v}\ \mathrm{d}S| \leq C\sqrt{h} + Ce^{\frac{R}{h}}\|\partial_\nu(\Delta(u_1-u_2))\|_{L^2(\partial\Omega_{-,\epsilon})}$$
$$+ \frac{C}{h}e^{\frac{R}{h}}\|\partial_\nu(u_1-u_2)\|_{L^2(\partial\Omega_{-,\epsilon})}.$$
$$\leq C\Big(\sqrt{h} + \frac{e^{\frac{R}{h}}}{h}\Big(\|\partial_\nu(\Delta(u_1-u_2))\|_{L^2(\partial\Omega_{-,\epsilon})}$$
$$+ \|\partial_\nu(u_1-u_2)\|_{L^2(\partial\Omega_{-,\epsilon})}\Big)\Big).$$
$$\leq C\sqrt{h} + \frac{C}{h}e^{\frac{R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|(\|u_2\|_{H^4(\Omega)} + \|\Delta u_2\|_{H^2(\Omega)})$$

and using the estimates (16) and (15), it follows that this is

$$\leq C\sqrt{h} + \frac{C}{h^5}e^{\frac{3R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|$$
$$\leq C\sqrt{h} + Ce^{\frac{8R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|,$$

where the constant $C$ now depends upon $\varepsilon$.

From the boundary Carleman estimate, we also have

$$h^{\frac{5}{2}}\|\sqrt{\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_{+,\varepsilon})} \leq C(\|e^{-\frac{x\cdot\alpha}{h}}(h^4\mathcal{B}_{q_1})u\|_{L^2(\Omega)}$$
$$+ h^{\frac{3}{2}}\|\sqrt{-\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(-h^2\Delta u)\|_{L^2(\partial\Omega_{-,\varepsilon})}$$
$$+ h^{\frac{5}{2}}\|\sqrt{-\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_{-,\varepsilon})})$$

and therefore

$$\|\sqrt{\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_{+,\varepsilon})} \leq C\left(h^{\frac{3}{2}}\|e^{-\frac{x\cdot\alpha}{h}}\mathcal{B}_{q_1}u\|_{L^2(\Omega)}\right.$$
$$+ h\|\sqrt{-\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta u)\|_{L^2(\partial\Omega_{-,\varepsilon})}$$
$$+ \left.\|\sqrt{-(\alpha\cdot\nu)}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_{-,\varepsilon})}\right).$$

We then have

$$\|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_{+,\epsilon})} \leq \frac{C}{\sqrt{\epsilon}}(h^{\frac{3}{2}}\|e^{-\frac{x\cdot\alpha}{h}}\mathcal{B}_{q_1}u\|_{L^2(\Omega)}$$
$$+ h\|\sqrt{-(\alpha\cdot\nu)}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta u)\|_{L^2(\partial\Omega_{-,\varepsilon})}$$
$$+ \|\sqrt{-(\alpha\cdot\nu)}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_{-,\varepsilon})})$$
$$\leq \frac{C}{\sqrt{\epsilon}}h^{\frac{3}{2}}\|e^{-\frac{x\cdot\alpha}{h}}(\mathcal{B}_{q_1})u\|_{L^2(\Omega)}$$
$$+ \frac{hC}{\sqrt{\epsilon}}\sqrt{-\inf_{\partial\Omega_-}(\alpha\cdot\nu)}\|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta u)\|_{L^2(\partial\Omega_{-,\epsilon})}$$
$$+ \frac{C}{\sqrt{\epsilon}}\sqrt{-\inf_{\partial\Omega_-}\alpha\cdot\nu}\|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_{-,\epsilon})}.$$

Similar to the previous estimate, we have

$$|\int_{\partial\Omega_{+,\epsilon}}\partial_\nu(u_1-u_2)\overline{(\Delta v)}\ \mathrm{d}S| \leq \frac{C}{\sqrt{\epsilon}}h^{\frac{3}{2}}\|e^{-\frac{x\cdot\alpha}{h}}(\mathcal{B}_{q_1})(u_1-u_2)\|_{L^2(\Omega)}$$
$$+ \frac{hC}{\sqrt{\epsilon}}\sqrt{-\inf_{\partial\Omega_-}\alpha\cdot\nu}\|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta(u_1-u_2))\|_{L^2(\partial\Omega_{-,\epsilon})}$$
$$+ \frac{C}{\sqrt{\epsilon}}\sqrt{-\inf_{\partial\Omega_-}(\alpha\cdot\nu)}\ \|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(u_1-u_2)\|_{L^2(\partial\Omega_{-,\epsilon})}$$
$$\leq Ch^{\frac{3}{2}} + \frac{C}{h^4}e^{\frac{3R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q2}\|$$
$$\leq Ch^{\frac{3}{2}} + Ce^{\frac{8R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q2}\|,$$

where the constant $C$ again depends upon $\varepsilon$.

Therefore using the estimates obtained above, we have

$$
\begin{aligned}
|\int_\Omega e^{-ix\cdot\xi}(q_2 - q_1)\,dx| &\leq |\int_{\partial\Omega_{+,\epsilon}} \partial_\nu(\Delta(u_1 - u_2))\bar{v}\,\mathrm{d}S| + |\int_{\partial\Omega_{+,\epsilon}} \partial_\nu(u_1 - u_2)\overline{(\Delta v)}\,\mathrm{d}S| \\
&\quad + |\int_{\partial\Omega_{-,\epsilon}} \partial_\nu(\Delta(u_1 - u_2))\bar{v}\,\mathrm{d}S + \int_{\partial\Omega_{-,\epsilon}} \partial_\nu(u_1 - u_2)\overline{(\Delta v)}\,\mathrm{d}S| \\
&\quad + |\int_\Omega (q_2 - q_1)e^{-ix\cdot\xi}(h\bar{r}_1 + hr_2 + h^2\bar{r}_1 r_2)\,dx| \\
&\leq C\sqrt{h} + Ce^{\frac{8R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| + Ch^{\frac{3}{2}} \\
&\quad + Ce^{\frac{8R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| \\
&\quad + Ce^{\frac{9R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| + Ch \\
\text{(19)} \qquad &\leq C(\sqrt{h} + e^{\frac{9R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|).
\end{aligned}
$$

The argument that now follows is similar to the one in [8]. We will apply Vessella's result given in Theorem 3.4 for the following set up. We take $D$ to be the ball $B(0,2)$ and $E = V \cap B(0,1)$ where $V$ is a suitable small open cone centered at $0$ obtained by perturbing the vector $\alpha$ slightly and recalling that $\xi$ is perpendicular to $\alpha$. Note that the above estimate is valid for all $\xi \in V$ such that $|\xi| < \frac{2}{h}$.

Now let $q = q_1 - q_2$ extended to $\mathbb{R}^n$ as $0$ outside $\Omega$ and for a fixed $\rho \in (0, \frac{2}{h})$, let $f(\xi) = \widehat{q}(\rho\xi)$. Then $f$ is analytic in $B(0,2)$ and

$$
|D^\alpha f(\xi)| \leq \|q\|_{L^1(\Omega)} \frac{\rho^{|\alpha|}}{(\mathrm{diam}(\Omega)^{-1})^{|\alpha|}} \leq 2M|\Omega||\alpha|! \frac{e^{n\rho}}{(\mathrm{diam}(\Omega)^{-1})^{|\alpha|}}.
$$

Taking $C$ and $\lambda$ in Vessella's result to be $C = 2M|\Omega|e^{n\rho}$ and $\lambda = \mathrm{diam}(\Omega)^{-1}$, we get that there exists a constant $\gamma_1 \in (0,1)$ such that

$$
|f(\xi)| \leq C^{1-\gamma_1(|E|/|D|)} \left(\sup_E |f(\xi)|\right)^{\gamma_1(|E|/|D|)}, \quad \text{for all } \xi \in B(0,1).
$$

Letting $\theta = \gamma_1|E|/|D|$, we have that for all $|\xi| < \rho$,

$$
\text{(20)} \qquad |\widehat{q}(\xi)| \leq C^{1-\theta}\left(\sup_{V\cap B(0,\rho)} |\widehat{q}(\xi/\rho)|\right)^\theta.
$$

Note that the constant $\theta$ is independent of $\rho$ and $h$. We have

$$
\begin{aligned}
\|q\|_{H^{-1}(\mathbb{R}^n)}^{\frac{2}{\theta}} &= \left(\int_{|\xi|<\rho} \frac{|\widehat{q}(\xi)|^2}{1+|\xi|^2}\mathrm{d}\xi + \int_{|\xi|\geq\rho} \frac{|\widehat{q}(\xi)|^2}{1+|\xi|^2}\mathrm{d}\xi\right)^{\frac{1}{\theta}} \\
&\leq C\left(\rho^{\frac{n}{\theta}}\|\widehat{q}\|_{L^\infty(B(0,\rho))}^{\frac{2}{\theta}} + \frac{1}{\rho^{\frac{2}{\theta}}}\right).
\end{aligned}
$$

The estimate of the second term on the right hand side above is obtained from Plancherel identity. Now from (20), it follows that the left hand side is

$$\leq C\left(\rho^{\frac{n}{\theta}}e^{2n\rho\frac{1-\theta}{\theta}}\|\widehat{q}(\xi)\|^2_{L^\infty(V\cap B(0,\rho))} + \frac{1}{\rho^{\frac{2}{\theta}}}\right)$$

$$\leq C\left(\rho^{\frac{n}{\theta}}e^{2n\rho\frac{1-\theta}{\theta}}e^{\frac{18R}{h}}\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|^2 + \rho^{\frac{n}{\theta}}e^{2n\rho\frac{1-\theta}{\theta}}h + \frac{1}{\rho^{\frac{2}{\theta}}}\right).$$

Let $L = \frac{3n+2-2n\theta}{\theta}$ and $\delta = e^{-e^{K/h_0^{\frac{1}{L}}}}$, where $K = \frac{2n+2}{\theta} + 4n\frac{1-\theta}{\theta} + 18R$. Let $\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| < \delta$.
Then choose $\rho = \frac{1}{K}\ln|\ln\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\||$ and

$$h = \frac{1}{\rho^{\frac{n+2}{\theta}}e^{\frac{2n\rho(1-\theta)}{\theta}}}.$$

*Claim 1:* $\rho < \frac{2}{h}$. We have $\rho h = \frac{\rho}{\rho^{\frac{n+2}{\theta}}e^{\frac{2n\rho(1-\theta)}{\theta}}} = \frac{1}{\rho^{\frac{n+2}{\theta}-1}e^{\frac{2n\rho(1-\theta)}{\theta}}} \leq \frac{1}{\rho^{\frac{n+2}{\theta}-1}}$.
Now since $n \geq 3$ we have $\frac{n+2}{\theta} - 1 > 4$, and hence $\rho^{\frac{n+2}{\theta}-1} > \rho^4$. Therefore, $\rho h < \frac{1}{\rho^4} < 2$ (since $\rho > 1$).

*Claim 2:* $h < h_0$. We have

$$\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| < e^{-e^{K/h_0^{\frac{1}{L}}}} \ll 1$$

$$\Rightarrow \ln\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| < -e^{K/h_0^{\frac{1}{L}}}$$

$$\Rightarrow |\ln\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|| > e^{K/h_0^{\frac{1}{L}}}$$

$$\Rightarrow \frac{1}{K}\ln|\ln\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|| > \frac{1}{h_0^{\frac{1}{L}}}$$

$$\Rightarrow \rho^L > \frac{1}{h_0}$$

$$\Rightarrow \rho^{\frac{3n+2-2n\theta}{\theta}} > \frac{1}{h_0}$$

$$\Rightarrow \frac{1}{\rho^{\frac{n+2}{\theta}}\rho^{\frac{2n(1-\theta)}{\theta}}} < h_0$$

$$\Rightarrow h = \frac{1}{\rho^{\frac{n+2}{\theta}}e^{\frac{2n\rho(1-\theta)}{\theta}}} \leq \frac{1}{\rho^{\frac{n+2}{\theta}}\rho^{\frac{2n(1-\theta)}{\theta}}} < h_0$$

*Claim 3:* $1 - h^2\frac{|\xi|^2}{4} > 0$. This is because $h^2\frac{|\xi|^2}{4} \leq h^2\frac{\rho^2}{4} = \frac{\rho^2}{4\rho^{\frac{2(n+2)}{\theta}}e^{4n\rho\frac{(1-\theta)}{\theta}}} =$
$\frac{\rho^{2-\frac{2n+4}{\theta}}}{4e^{4n\rho\frac{1-\theta}{\theta}}} < \rho^{2-\frac{2n+4}{\theta}} = \frac{1}{\rho^{\frac{2n+4}{\theta}-2}}$. Now $\frac{2n+4}{\theta} - 2 > 8$ since $n \geq 3$ and therefore

$\rho^{\frac{2n+4}{\theta}-2} \geq \rho^8$ which in turn implies that

$$h^2 \frac{|\xi|^2}{4} < \frac{1}{\rho^8} < 1.$$

Then we have

$$\rho^{\frac{n}{\theta}} e^{2n\rho \frac{1-\theta}{\theta} + \frac{18R}{h}} = \rho^{\frac{n}{\theta}} e^{[2n\rho \frac{1-\theta}{\theta} + 18R(\rho^{\frac{n+2}{\theta}} e^{2n\rho \frac{1-\theta}{\theta}})]}$$

$$\leq e^{\frac{n}{\theta}\rho + 2n\rho \frac{1-\theta}{\theta} + 18R\rho^{\frac{n+2}{\theta}} e^{2n\rho \frac{1-\theta}{\theta}}} \text{ since } \rho^{\frac{n}{\theta}} \leq e^{\frac{n}{\theta}\rho}$$

$$\leq e^{e^{\left(\frac{n}{\theta}\rho + 2n\rho \frac{1-\theta}{\theta}\right)} + e^{\left(18R + \frac{n+2}{\theta}\rho + 2n\rho \frac{1-\theta}{\theta}\right)}}$$

$$\leq e^{e^{\left(\frac{n}{\theta}\rho + 2n\rho \frac{1-\theta}{\theta}\right)} + e^{\left(18R\rho + \frac{n+2}{\theta}\rho + 2n\rho \frac{1-\theta}{\theta}\right)}} \text{ since } \rho \geq 1$$

$$\leq C e^{e^{\left(\frac{n}{\theta} + 2n\frac{1-\theta}{\theta} + 18R + \frac{n+2}{\theta} + 2n\frac{1-\theta}{\theta}\right)\rho}} \text{ since } e^a + e^b \leq 1 + e^{a+b}.$$

Therefore since $K = \frac{2n+2}{\theta} + 4n\frac{1-\theta}{\theta} + 18R$ and $\rho = \frac{1}{K} \ln |\ln \|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\||$, we obtain

$$\|q\|_{H^{-1}(\Omega)}^{\frac{2}{\theta}} \leq C(\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| + \ (\frac{1}{K} \ln |\ln \|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\||)^{-\frac{2}{\theta}})$$

and hence

$$\|q\|_{H^{-1}(\Omega)} \leq C(\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| + \ (\frac{1}{K} \ln |\ln \|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\||)^{-\frac{2}{\theta}})^{\frac{\theta}{2}},$$

whenever $\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| < \delta$.
When $\|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\| \geq \delta$, we have

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C\|q_1 - q_2\|_{L^\infty(\Omega)} \leq \frac{2CM}{\delta^{\frac{\theta}{2}}} \delta^{\frac{\theta}{2}} \leq \frac{2CM}{\delta^{\frac{\theta}{2}}} \|\widetilde{\mathcal{N}}_{q_1} - \widetilde{\mathcal{N}}_{q_2}\|^{\frac{\theta}{2}}$$

and the desired estimate follows. $\qquad\square$

## APPENDIX A.

In this section we prove the unique determination of $q$ from (2) when the Neumann data $\widetilde{N}_q$ is known on slightly more than half the boundary. This is already done in a more general set-up with limiting Carleman weights in [11], where the authors use logarithmic weights. We give here the proof with linear Carleman weight following [2] for the sake of completeness.

**Theorem A.1.** [11] *Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with smooth boundary. Consider Equation (2) for two potentials $q_1, q_2 \in L^\infty(\Omega)$. Let $\widetilde{\mathcal{N}}_{q_1}$ and $\widetilde{\mathcal{N}}_{q_2}$ be the corresponding Dirichlet-to-Neumann maps measured on $\partial\Omega_{-,\varepsilon}$. If $\widetilde{N}_{q_1} = \widetilde{N}_{q_2}$, then $q_1 = q_2$.*

*Proof.* As before, we start with the following integral identity.

$$\int_\Omega (q_2 - q_1) u_2 \bar{v} \, dx = -\int_{\partial\Omega_{+,\epsilon}} \partial_\nu(-\Delta(u_1 - u_2))\bar{v} \, \mathrm{d}S - \int_{\partial\Omega_{+,\epsilon}} \partial_\nu(u_1 - u_2)\overline{(-\Delta v)} \, \mathrm{d}S$$

$$|\int_{\partial\Omega_{+,\epsilon}} \partial_\nu(\Delta(u_1-u_2))\bar{v}\ \mathrm{d}S| = |\int_{\partial\Omega_{+,\epsilon}} e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta(u_1-u_2))e^{\frac{x\cdot\alpha}{h}}\bar{v}\ \mathrm{d}S|$$

$$\leq \|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta(u_1-u_2))\|_{L^2(\partial\Omega_{+,\epsilon})}\|e^{\frac{x\cdot\alpha}{h}}\bar{v}\|_{L^2(\partial\Omega_{+,\epsilon})}$$

$$|\int_{\partial\Omega_{+,\epsilon}} \partial_\nu(u_1-u_2)\overline{(\Delta v)}\ \mathrm{d}S| = |\int_{\partial\Omega_{+,\epsilon}} e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(u_1-u_2)e^{\frac{x\cdot\alpha}{h}}\overline{(\Delta v)}\ \mathrm{d}S|$$

$$\leq \|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(u_1-u_2)\|_{L^2(\partial\Omega_{+,\epsilon})}\|e^{\frac{x\cdot\alpha}{h}}\overline{(\Delta v)}\|_{L^2(\partial\Omega_{+,\epsilon})}$$

From the boundary Carleman estimate, we have

$$\|\sqrt{\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\ \partial_\nu(-h^2\Delta u)\|_{L^2(\partial\Omega_+)} \leq \frac{C}{h^{\frac{3}{2}}}\|e^{-\frac{x\cdot\alpha}{h}}(h^4\mathcal{B}_{q_1})u\|_{L^2(\Omega)}$$

where $u = u_1 - u_2$.
Using this we get

$$\sqrt{\epsilon}\|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(-h^2\Delta u)\|_{L^2(\partial\Omega_{+,\epsilon})} \leq \|\sqrt{\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(-h^2\Delta u)\|_{L^2(\partial\Omega_{+,\epsilon})}$$

$$\leq \|\sqrt{\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(-h^2\Delta u)\|_{L^2(\partial\Omega+)}$$

$$\leq \frac{C}{h^{\frac{3}{2}}}\|e^{-\frac{x\cdot\alpha}{h}}(h^4\mathcal{B}_{q_1})u\|_{L^2(\Omega)}.$$

Therefore

$$\|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(\Delta(u_1-u_2))\|_{L^2(\partial\Omega_{+,\epsilon})} \leq C\sqrt{\frac{h}{\epsilon}}\|e^{-\frac{x\cdot\alpha}{h}}(q_2-q_1)e^{\frac{\mathrm{i}x\cdot\zeta_2}{h}}(1+hr_2)\|_{L^2(\Omega)}$$

$$\leq C\sqrt{\frac{h}{\epsilon}}\|(1+hr_2)\|_{L^2(\Omega)} \leq C\sqrt{\frac{h}{\epsilon}}.$$

From the boundary Carleman estimate, we also have

$$\|\sqrt{\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_+)} \leq \frac{C}{h^{\frac{5}{2}}}\|e^{-\frac{x\cdot\alpha}{h}}(h^4\mathcal{B}_{q_1})u\|_{L^2(\Omega)}$$

Again using this, we get

$$\sqrt{\epsilon}\|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_{+,\epsilon})} \leq \|\sqrt{\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_{+,\epsilon})}$$

$$\leq \|\sqrt{\alpha\cdot\nu}\ e^{-\frac{x\cdot\alpha}{h}}\partial_\nu u\|_{L^2(\partial\Omega_+)}$$

$$\leq \frac{C}{h^{\frac{5}{2}}}\|e^{-\frac{x\cdot\alpha}{h}}(h^4\mathcal{B}_{q_1})u\|_{L^2(\Omega)}$$

$$\Rightarrow \|e^{-\frac{x\cdot\alpha}{h}}\partial_\nu(u_1-u_2)\|_{L^2(\partial\Omega_{+,\epsilon})} \leq \frac{C}{h^{\frac{5}{2}}\sqrt{\epsilon}}\|e^{-\frac{x\cdot\alpha}{h}}(h^4\mathcal{B}_{q_1})u\|_{L^2(\Omega)}$$

$$\leq \frac{Ch^{\frac{3}{2}}}{\sqrt{\epsilon}}\|(1+hr_2)\|_{L^2(\Omega)} \leq \frac{Ch^{\frac{3}{2}}}{\sqrt{\epsilon}}$$

Next we show that the terms $\|e^{\frac{x\cdot\alpha}{h}}\bar{v}\|_{L^2(\partial\Omega_{+,\epsilon})}$ and $\|e^{\frac{x\cdot\alpha}{h}}\overline{(\Delta v)}\|_{L^2(\partial\Omega_{+,\epsilon})}$ are bounded. The term

$$\|e^{\frac{x\cdot\alpha}{h}}\bar{v}\|_{L^2(\partial\Omega_{+,\epsilon})} \leq \|e^{\frac{x\cdot\alpha}{h}}\bar{v}\|_{L^2(\partial\Omega)} = \|(1+h\overline{r_1})\|_{L^2(\partial\Omega)}$$

$$\leq C(1+\|hr_1\|_{H^1(\Omega)}) \leq C,$$

since $h \ll 1$.
Again

$$e^{\frac{x \cdot \alpha}{h}}\overline{(\Delta v)} = e^{\frac{i}{h}(-\frac{h}{2}(x \cdot \xi) - \sqrt{1 - h^2 \frac{|\xi|^2}{4}}(x \cdot \beta))}h\Delta\overline{r_1} - 2ie^{\frac{i}{h}(-\frac{h}{2}(x \cdot \xi) - \sqrt{1 - h^2 \frac{|\xi|^2}{4}}(x \cdot \beta))}\overline{\zeta_1} \cdot \nabla \overline{r_1}.$$

Therefore, we have

$$\begin{aligned}
\|e^{\frac{x \cdot \alpha}{h}}\overline{(\Delta v)}\|_{L^2(\partial\Omega_{+,\epsilon})} &\leq \|e^{\frac{x \cdot \alpha}{h}}\overline{(\Delta v)}\|_{L^2(\Omega)} \\
&\leq \|h\Delta r_1\|_{L^2(\partial\Omega)} + C\|\nabla r_1\|_{L^2(\partial\Omega)} \\
&\leq C(h\|\Delta r_1\|_{H^1(\Omega)} + \|\nabla r_1\|_{H^1(\Omega)}) \\
&\leq C(\frac{h}{h^2}\|r_1\|_{H^4_{\mathrm{scl}}(\Omega)} + \frac{1}{h}\|r_1\|_{H^4_{\mathrm{scl}}(\Omega)}) \leq C.
\end{aligned}$$

Also using the estimates on $r_1, r_2$ it follows that as limit $h \to 0$,

$$\int_\Omega (q_2 - q_1)u_2\bar{v} \; dx \to \int_\Omega e^{-ix \cdot \xi}(q_2 - q_1) \; dx.$$

Therefore combining all the above estimates and passing to the limit as $h \to 0$, we have

$$\int_\Omega e^{-ix \cdot \xi}(q_2 - q_1) \; dx = 0$$

for all $\xi \in \mathbb{R}^n$ perpendicular to $\alpha$. Varying $\alpha$ in a sufficiently small neighborhood, we see that above estimates is true for all $\xi$ in an open cone in $\mathbb{R}^n$. A simple application of the Paley-Wiener theorem then implies that $q_2 = q_1$ on $\Omega$. This concludes the proof. $\qquad\square$

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